Spring 2016 Math 245 Main Midterm Solutions

Problem 1. Carefully state the "Division Algorithm" theorem.

For any integers a, b, with $b \geq 1$, there are unique integers q, r with $a = bq + r$ and $0 \leq r < b$.

Problem 2. Prove that the square of a rational number is rational.

Let $x \in \mathbb{Q}$. Then there are $m, n \in \mathbb{Z}$, with $n \neq 0$, such that $x = \frac{m}{n}$. Now $x^2 = \frac{m^2}{n^2}$. Also $m^2, n^2 \in \mathbb{Z}$, and $n^2 \neq 0$ since $n \neq 0$. Hence $x^2 \in \mathbb{Q}$.

Problem 3. Let $n \in \mathbb{Z}$. Prove that $\lceil \frac{n}{2} \rceil \geq \frac{n-1}{2}$.

Whoops! I meant to have $\lfloor \frac{n}{2} \rfloor \geq \frac{n-1}{2}$, which requires proof by cases. With the question as written, the solution is much simpler: $\lceil \frac{n}{2} \rceil \geq \frac{n}{2} > \frac{n-1}{2}$.

Problem 4. Carefully define each of the following terms:

a. nand

Nand is a symbolic connective in propositional calculus, which is false exactly when both components are true, and true otherwise.

b. hypothetical syllogism

Hypothetical syllogism is a rule of inference which concludes $p \to q$ from the hypotheses $p \to r$ and $r \to q$.

c. constructive existence proof

A constructive existence proof of the proposition $\exists x \in D,\ P(x)$ is made by explicitly finding some $x \in D$ such that $P(x)$ holds.

d. $\lfloor x \rfloor$

The floor of x, denoted $\lfloor x \rfloor$, is the greatest integer less than or equal to x.

e. proof by contradiction

A proof by contradiction of a proposition P is done by assuming that P does not hold, and deriving a contradiction from that hypothesis.

Problem 5. Carefully define each of the following terms:

a. strong induction

We prove $\forall x \in \mathbb{N},\ P(x)$ by strong induction by (i) proving the base case $P(1)$, and (ii) by assuming $P(1), P(2), \ldots, P(n)$ all hold and deriving $P(n+1)$.

b. $a|b$ $(a, b \in \mathbb{Z})$

We say a divides b (denoted $a|b$) to mean that there is some $c \in \mathbb{Z}$ with $b = ac$.

c. $A \subseteq B$

We say A is a subset of B (denoted $A \subseteq B$) if every element of A is an element of B.

d. $A \cap B$

The intersection of A and B (denoted $A \cap B$) is the set that contains all elements in both A and B.

e. $|A|$ (A is a set)

The cardinality of A (denoted $|A|$) is the number of elements in A.

Problem 6. Prove or disprove: $\forall x \in \mathbb{R}, \exists y, z \in \mathbb{R},\ y^2 < x^2 < z^2$.

The statement is false. We disprove with the counterexample $x = 0$. Now, for all $y, z \in \mathbb{R}$, $y^2 < x^2 < z^2$ fails to hold, because $y^2 < x^2 = 0$ cannot hold for any real number y.

Problem 7. Give a mathematical statement with one free variable and two bound variables.

Many solutions are possible. Two examples related to problem 6 are: (i) $\exists y, z \in \mathbb{R}, y^2 < x^2 < z^2$, and (ii) $\forall x \in \mathbb{R} \exists y \in \mathbb{R}, y^2 < x^2 < z^2$.

Problem 8. A Boolean algebra is a nonempty set S, two binary operations $\oplus, \odot$, and six axioms. Carefully state any three of these axioms.

The six axioms are found on p. 118 of the text, at the beginning of Chapter 14.

Problem 9. Let A, B be sets. Prove that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

Let $x \in \mathcal{P}(A) \cup \mathcal{P}(B)$. We have two cases: (i) $x \in \mathcal{P}(A)$. Hence $x \subseteq A \subseteq A \cup B$. Hence $x \in \mathcal{P}(A \cup B)$. (ii) $x \in \mathcal{P}(B)$. Hence $x \subseteq B \subseteq A \cup B$. Hence $x \in \mathcal{P}(A \cup B)$.

Problem 10. Prove that $\sqrt{3}$ is irrational.

Arguing by contradiction, we assume that $\sqrt{3} = \frac{m}{n}$, where $m, n \in \mathbb{Z}$ and have no common factors. We square both sides to get $m^2 = 3n^2$. Hence $3|m^2$. Since 3 is prime, $3|m$. We write $m = 3k$, for some integer k. We have $3n^2 = m^2 = (3k)^2 = 9k^2$. Cancelling 3, we get $n^2 = 3k^2$. Hence $3|n^2$. Since 3 is prime, $3|n$. This contradicts the hypothesis that m, n have no common factors.

Problem 11. Use the extended Euclidean algorithm to find $\gcd(56, 133)$ and to find integers x, y so that $\gcd(56, 133) = 56x + 133y$.

We begin with $133 = 2 \cdot 56 + 21$, $56 = 2 \cdot 21 + 14$, $21 = 1 \cdot 14 + 7$, $14 = 2 \cdot 7 + 0$. Hence $\gcd(56, 133) = 7$. We now work backwards as $7 = 21 - 1 \cdot 14 = 21 - 1 \cdot (56 - 2 \cdot 21) = 3 \cdot 21 - 1 \cdot 56 = 3 \cdot (133 - 2 \cdot 56) - 1 \cdot 56 = 3 \cdot 133 - 7 \cdot 56$. Hence $x = -7, y = 3$.

Problem 12. Use induction to prove that $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ for all $n \in \mathbb{N}$.

Base case $n = 1$: LHS=$\frac{1}{1\cdot 2}$, RHS=$\frac{1}{1+1}$, which agree. Inductive case: Now assume that $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ holds for some $n \geq 1$. We add the same term to both sides, namely $\frac{1}{(n+1)(n+2)}$. The RHS is $\frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n(n+2)+1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2}$. Hence we have proved $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2}$, as desired.